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On the Congruence of Axes in a Bundle of Linear Line Complexes.

By O. P. Akers.

Introduction.

The congruence of axes contained in a three term group of linear line complexes has been quite exhaustively studied. Stahl* has treated in a synthetic manner, the (3, 2) congruence in considerable detail. Later† he discusses this congruence as a special case of a congruence of a higher order. We will take occasion later to refer to other papers on this subject. The discussion here will be with regard to some of the ruled surfaces contained in the congruence of axes. Some of the notations and methods of Study‡ will be used and some applications of these methods will be made to show the arrangement of the lines of the congruence on ruled surfaces contained in a linear complex.

If we choose the complex $\phi_1 = 0$ arbitrarily, $\phi_2 = 0$ in involution with it, and finally $\phi_3 = 0$ in involution with both, the coordinate complexes of the bundle may be written

$$p_{23} - e_1 p_{41} = 0, \qquad p_{31} - e_2 p_{42} = 0, \qquad p_{12} - e_3 p_{43} = 0,$$

and the bundle of complexes may be expressed by the equation

$$\sigma_1 (p_{23} - e_1 p_{41}) + \sigma_2 (p_{31} - e_2 p_{42}) + \sigma_3 (p_{12} - e_3 p_{43}) = 0. \tag{1}$$

This equation defines ∞^2 complexes by varying the $\sigma_1, \sigma_2, \sigma_3$. Moreover, ∞^1 coaxial complexes, for given values of the σ_i, are defined by the equation

$$\sigma_1 (p_{23} - e_1 p_{41}) + \sigma_2 (p_{31} - e_2 p_{42}) + \sigma_3 (p_{12} - e_3 p_{43}) + p(\sigma_1 p_{41} + \sigma_2 p_{42} + \sigma_3 p_{43}) = 0, \tag{2}$$

by giving p different values.

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* "Das Strahlensystem dritter Ordnung und zweiter Klasse"; Crelle, vol. 91, p. 1.

† "Das Strahlensystem vierter Ordnung und zweiter Klasse"; Crelle, vol. 97, p. 147.

‡ Geometrie der Dynamen; p. 461.

Since the axis is the same for every value of p, there is no restriction in taking the finite special complex to define the axis. If we solve the equation (2) for the coordinates of the axis, we obtain, when $e_1+e_2+e_3=0$ and $p(\sigma_1^2+\sigma_2^2+\sigma_3^2)=e_1\sigma_1^2+e_2\sigma_2^2+e_3\sigma_3^2$, the coordinates

$$p_{41}=\sigma_1, \qquad p_{42}=\sigma_2, \qquad p_{43}=\sigma_3,$$
$$p_{23}=(p-e_1)\sigma_1, \qquad p_{31}=(p-e_2)\sigma_2, \qquad p_{12}=(p-e_3)\sigma_3.$$

It will be assumed that $e_1>e_2>e_3$. This system of axes will be referred to as the (σ) system. The reciprocal (τ) system, defined by the system of three terms, every complex of which is in involution with every complex of the original bundle, has the coordinates

$$r_{41}=\tau_1, \qquad r_{42}=\tau_2, \qquad r_{43}=\tau_3,$$
$$r_{23}=-(p-e_1)\tau_1, \qquad r_{31}=-(p-e_2)\tau_2, \qquad r_{12}=-(p-e_3)\tau_3.$$

The (σ) system and the (τ) system will enter into the discussion in exactly the same way.

The σ_i may be regarded as point coordinates in the plane at infinity; and in this way the configuration can be mapped on the plane. To every point σ_i in the plane at infinity corresponds one finite line of the congruence and conversely, when a finite line is given, the values of σ_i are uniquely determined. The points of the plane at infinity and the lines of the congruence are in (1, 1) correspondence.

The congruence of axes in a bundle of linear line complexes contains fifteen singular planes and ten singular points. The dual case was first studied by Kummer ("Ueber die algebraischen Strahlensysteme, in's besondere ueber die der ersten und zweiten Ordnung." Berliner Abhandlungen, 1866). Five singular planes, one of which is the plane at infinity, are of the second order, the remaining ten are of the first order. Four singular points lie in each singular plane. The singular points on the absolute will be called cusps. The planes of the singular pencils which contain the tangents to the absolute at the cusps will be called cuspidal planes. The remaining six singular points will be called vertices, and the plane containing the pencil of lines passing through a vertex will be called a vertical plane.

The configuration in this congruence of axes is a particular case of the general (3, 2) congruence, the dual of the (2, 3) congruence discussed by Kummer. In the former case, four of the singular conics touch the fifth (the absolute), but in the latter case this is not, in general, true.

The coordinates of the cusps may be obtained from the relation $p(\sigma_1^2+\sigma_2^2+\sigma_3^2)=e_1\sigma_1^2+e_2\sigma_2^2+e_3\sigma_3^2$. When $\sigma_1^2+\sigma_2^2+\sigma_3^2=0$, and $e_1\sigma_1^2+e_2\sigma_2^2+e_3\sigma_3^2=0$, p is indeterminate. The coordinates of the common points of the two conics are

$$\sigma_1:\sigma_2:\sigma_3=\sqrt{e_2-e_3}:\sqrt{e_3-e_1}:\sqrt{e_1-e_2},$$

the signs of the radicals being permuted for the different points. Through these four points pass ∞^1 axes lying in a plane. If we write the equation of the plane determined by two lines of the congruence which pass through a cusp, we obtain the following,

$$s:t:u:v=\sqrt{e_2-e_3}\sqrt{e_3-e_1}\sqrt{e_1-e_2}:\sqrt{e_2-e_3}:\sqrt{e_3-e_1}:\sqrt{e_1-e_2},$$

which are the coordinates of the cuspidal planes in the (σ) congruence. The corresponding cuspidal planes in the (τ) congruence have the coordinates

$$s:t:u:v=-\sqrt{e_2-e_3}\sqrt{e_3-e_1}\sqrt{e_1-e_2}:\sqrt{e_2-e_3}:\sqrt{e_3-e_1}:\sqrt{e_1-e_2}.$$

If the coordinates of the cusp C_1 are $-\sqrt{e_2-e_3}:\sqrt{e_3-e_1}:\sqrt{e_1-e_2}$ and those of C_2 are $\sqrt{e_2-e_3}:-\sqrt{e_3-e_1}:\sqrt{e_1-e_2}$, then the equation of the line joining C_1 and C_2 is $\sqrt{e_1-e_2}\,\sigma_1-\sqrt{e_2-e_3}\,\sigma_3=0$. To every point of the line C_1C_2 corresponds one finite line of the congruence. All these form a plane pencil in some plane passing through C_1C_2. To find the plane and the vertex of the pencil, find the lines corresponding to any two points on C_1C_2. These two lines have a plane and a point in common. The equations of this plane and point are found to be

$$\sqrt{e_1-e_2}\,x+\sqrt{e_2-e_3}\,z=0, \qquad s+\sqrt{e_1-e_2}\sqrt{e_2-e_3}\,u=0,$$

in the (σ) congruence or

$$\sqrt{e_1-e_2}\,x+\sqrt{e_2-e_3}\,z=0, \qquad s-\sqrt{e_1-e_2}\sqrt{e_2-e_3}\,u=0,$$

in the (τ) congruence. There are two real vertices and vertical planes. The six vertical planes intersect by twos, on the principal rays, lines passing through the vertices of the common self conjugate triangle with reference to the absolute and the curve $e_1\sigma_1^2+e_2\sigma_2^2+e_3\sigma_3^2=0$. These three lines are mutually perpendicular and pass through a common point. It is convenient to take this point for origin (0, 0, 0, 1). Each principal ray contains two vertices. Each cuspidal plane contains one cusp and three vertices, and each vertical plane contains two cusps and two vertices. The vertices and planes are the same for both congruences, but differently paired. The arrangement of the singular points and the singular planes is readily seen from the tables. C_i are the cusps; c_i are the corresponding cuspidal planes. S_i, S_i' are the vertices lying on the principal ray i; s_i, s_i' are the vertical planes intersecting on i. Each singular plane contains the four singular points of the same column.

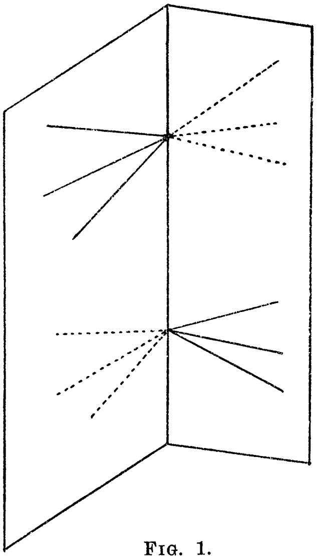

FIG. 1.

c_1	c_2	c_3	c_4
C_1	C_2	C_3	C_4
S_1	S_1'	S_1	S_1'
S_2	S_2	S_2'	S_2'
S_3'	S_3	S_3'	S_3

s_1	s_1'	s_2	s_2'	s_3	s_3'
C_1	C_3	C_1	C_2	C_1	C_2
C_2	C_4	C_3	C_4	C_4	C_3
S_1	S_1	S_2	S_2	S_3	S_3
S_1'	S_1'	S_2'	S_2'	S_3'	S_3'

CHAPTER I.

§1. *Parameter Surfaces.*

The equations of any line may be written

$$\left.\begin{aligned} p_{43}\,y + p_{24}\,z + p_{32}\,w = 0 \\ p_{34}\,x + p_{41}\,z + p_{13}\,w = 0. \end{aligned}\right\} \tag{1}$$

When p is defined by $p(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) = e_1\sigma_1^2 + e_2\sigma_2^2 + e_3\sigma_3^2$, any line of the congruence may be expressed by

$$\left.\begin{aligned} \sigma_3\,y - \sigma_2\,z - (p - e_1)\,\sigma_1\,w = 0 \\ -\sigma_3\,x + \sigma_1\,z - (p - e_2)\,\sigma_2\,w = 0. \end{aligned}\right\} \tag{2}$$

If the last equations be solved for $\sigma_1 : \sigma_2 : \sigma_3$, the result is

$$\frac{\sigma_1}{xz + a_2\,yw} = \frac{\sigma_2}{yz - a_1\,xw} = \frac{\sigma_3}{z^2 + a_1 a_2 w^2}, \tag{3}$$

where $a_i = p - e_i$. Substituting the value of p in equation (3), we have the equation of the parameter surface

$$a_1 x^2 + a_2 y^2 + a_3 z^2 + a_1 a_2 a_3 w^2 = 0, \tag{4}$$

or in plane coordinates

$$a_2 a_3 s^2 + a_3 a_1 t^2 + a_1 a_2 u^2 + v^2 = 0. \tag{5}$$

One system of lines on this surface belongs to the (σ) congruence and the other system belongs to the (τ) congruence for every value of p.[*] To each value of p corresponds a parameter surface. The equation

$$p(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) = e_1 \sigma_1^2 + e_2 \sigma_2^2 + e_3 \sigma_3^2$$

represents a pencil of conics in the plane $w = 0$, having the cusps for basis points. One point will fix a conic; the same point fixes a finite line of the congruence, hence every line of the congruence lies on just one parameter surface. For $p = \infty$, $\sigma_1^2 + \sigma_2^2 + \sigma_3^2 = 0$, hence the absolute is included as a degraded case of a parameter surface. The plane $w = 0$, is therefore a singular plane of the second order; through any point in it pass two tangents to the absolute and one finite line. Between the latter and the point exist a (1, 1) correspondence. For points on the absolute, the line is also tangent to the absolute, and at the cusps the correspondence breaks down, as every line of the cuspidal pencil corresponds to the same point, namely, the cusp. If the coordinates of an arbitrary point (x', y', z', w') be substituted in equation (4), we obtain a cubic equation in p, which shows that three parameter surfaces of the system can be drawn through each point of space. The tangent planes at (x', y', z', w') each contain two generators, one generator belongs to the (σ) system and the other belongs to the (τ) system. Since a line of (σ) uniquely fixes the parameter surface to which it belongs, through each point of space pass three lines of the congruence and the congruence is of the order three. The equation of the parameter surface (4) may be factored into two expressions and written

$$\left.\begin{aligned} \lambda(\sqrt{a_2}\, y + i\sqrt{a_3}\, z) &= \sqrt{a_1 a_2 a_3}\, w + i\sqrt{a_1}\, x \\ \lambda(\sqrt{a_1 a_2 a_3}\, w - i\sqrt{a_1}\, x) &= -\sqrt{a_2}\, y + i\sqrt{a_3}\, z \end{aligned}\right\} \qquad \text{A}$$

or

$$\left.\begin{aligned} \mu(\sqrt{a_2}\, y - i\sqrt{a_3}\, z) &= \sqrt{a_1 a_2 a_3}\, w + i\sqrt{a_1}\, x \\ \mu(\sqrt{a_1 a_2 a_3}\, w - i\sqrt{a_1}\, x) &= -\sqrt{a_2}\, y - i\sqrt{a_3}\, z. \end{aligned}\right\} \qquad \text{B}$$

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[*] See Waelsch, "Ueber eine Strahlencongruenz beim Hyperboloid." Wiener Berichte, vol. 95, pp. 781–801, 1887; also Stahl, Crelle, vol. 91.

If (x_1, y_1, z_1, w_1) are the coordinates of a fixed point lying on the hyperboloid, we may eliminate λ and μ from these two groups of equations and get the relations

$$\begin{aligned}\sqrt{a_1 a_2 a_3}\,(\sqrt{a_2}\,y_1 + i\sqrt{a_3}\,z_1)\,w + i\sqrt{a_1}\,(\sqrt{a_2}\,y_1 + i\sqrt{a_3}\,z_1)\,x \\ -(a_2\sqrt{a_1 a_3}\,w_1 + i\sqrt{a_2 a_1}\,x_1)\,y - (ia_3\sqrt{a_1 a_2}\,w_1 - \sqrt{a_1 a_3}\,x_1)\,z = 0,\end{aligned}$$

and

$$\begin{aligned}\sqrt{a_2}\,(\sqrt{a_1 a_2 a_3}\,w_1 - i\sqrt{a_1}\,x_1)\,y - i\sqrt{a_3}\,(\sqrt{a_1 a_2 a_3}\,w_1 - i\sqrt{a_1}\,x_1)\,z \\ -\sqrt{a_1 a_2 a_3}\,(\sqrt{a_2}\,y_1 - i\sqrt{a_3}\,z_1)\,w + i\sqrt{a_1}\,(\sqrt{a_2}\,y_1 - i\sqrt{a_3}\,z_1)\,x = 0.\end{aligned}$$

If we express the line coordinates from these last two equations, we have

$$\begin{aligned}\sigma_1 ; \sigma_2 : \sigma_3 = x^2 + a_2 a_3 w^2 : xy + a_3 wz : xz - a_2 wy = xy - a_3 wz : \\ y^2 + a_3 a_1 w^2 : yz + a_1 wx = xz + a_2 wy : yz - a_1 wx : z^2 + a_1 a_2 w^2, \qquad (6)\end{aligned}$$

which shows the relation among the σ_i for an arbitrary point (x, y, z, w). If this point lies in the plane $w = 0$, but not on the absolute, then two roots of equation (4) become infinite. If the point is on the absolute but not at a cusp, all three roots are infinite. If however, the point is at a cusp, p is indeterminate and (6) furnishes a pencil of lines through this point. When the coordinates of the real vertices, $1 : 0 : \mp\sqrt{e_1 - e_2}\,\sqrt{e_2 - e_3} : 0$, are substituted we get the lines of the vertical pencils. Similarly, equation (5) gives the relation

$$\begin{aligned}\sigma_1 : \sigma_2 : \sigma_3 = -a_3 u^2 - a_2 v^2 : a_3 tu + sv : a_2 tv - su = a_3 tu - sv : \\ -a_1 v^2 - a_3 t^2 : a_1 uv + st = a_2 tv + su : a_1 uv - st : -a_2 t^2 - a_1 u^2. \qquad (7)\end{aligned}$$

If the plane (s, t, u, v) touches the absolute, but not at a cusp, then equation (5) has one infinite root; but if the point of contact is a cusp and the plane is not a cuspidal plane, both roots are infinite. If the plane is a cuspidal plane of (τ), (7) furnishes ∞^1 lines of (σ) which envelope a conic, which touches the absolute, the tangent at the cusp being the common tangent.

§2. *Focal Surface.*

The locus of the ∞^2 points of intersection of consecutive lines of the congruence is called the focal surface. Evidently the equation of this locus may be obtained by considering the condition that the equation of the parameter surface

(4), as a cubic in p, shall have two equal roots.* Hence writing the discriminant of that equation we have

$$4\{(e_2 e_3 + e_3 e_1 + e_1 e_2) w^2 + x^2 + y^2 + z^2\}^3 + 27w^2 (e_1 e_2 e_3 w^2 + e_1 x^2 + e_2 y^2 + e_3 z^2)^2 = 0. \tag{8}$$

Similarly equation (5) gives

$$4(t^2 + u^2 + v^2)(s^2 + e_2 e_3 t^2 + e_3 e_1 u^2 + e_1 e_2 v^2) - (e_1 t^2 + e_2 u^2 + e_3 v^2)^2 = 0, \tag{9}$$

which shows that the focal surface is of order six and class four, which agrees with the results obtained by Kummer's formulas.

The focal surface is the same for both congruences. It is the complete envelope of the parameter surfaces, and it is also the locus of the limiting points and limiting planes. Each point of the surface is a focal point for one line, *i. e.*, two consecutive lines of the congruence and a limiting point for another line of the congruence. Each plane of the focal surface is a focal plane for one line, *i. e.*, two consecutive lines of the congruence, and a limiting plane for another line of the congruence, namely, for the third line of the congruence which passes through the focus. The intersection of the $(p = 0)$ hyperboloid with its director sphere is a cuspidal curve on the focal surface.† Moreover, the plane $w = 0$ intersects the focal surface in the absolute, which is also a cuspidal curve. These curves are the loci of points for which the tangent planes to the focal surface and the tangent planes to the parameter surface $p = 0$, coincide; *i. e.*, from points on these curves passes only one line of the congruence counted three times. Since in the plane $w = 0$, the tangents to the absolute are lines of the congruence, the focal points and limiting points coincide on the absolute and we have lines of six point contact.

From the equation of the focal surface, in plane coordinates, it is seen that a cuspidal plane is tangent to the surface in a conic C_1, which is the locus of the point of contact of this plane with the parameter surface. Moreover, the same cuspidal plane intersects the focal surface in another conic C_2. The two conics have a common tangent, namely, the tangent to the absolute at a cusp. We have seen that the cuspidal plane contains a pencil of minimal lines which belongs to (σ), and whose vertex is the cusp; and that it contains, also, ∞^1 finite,

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[*] Hyde, "On a surface of the sixth order which is touched by the axes of all screws reciprocal to three given screws." *Ann. Math.*, series 2, vol. 2, p. 179.

Gruenwald, "Zur Veranschaulichung des Schraubenbuendels" (*Schloemilch's Zeitschrift*, vol. 49, p. 211).

Joly, "Geometry of a three system of screws." *Trans. Royal Acad.*, vol. 32, Sec. A, pp. 239–270.

[†] Stahl, *Crelle*, vol. 97, p. 146, sec. 8.

but imaginary lines of (τ) which envelope the conic C_2. The focal points of the minimal lines lie on the conic C_1. Their limiting points are on C_2. At the cusp, one focal point and a limiting point coincide. It is interesting to notice that the (τ) lines, in this plane, touch the focal surface in three points. However these are not all three focal points, but at one of the points of tangency the limiting points have come into coincidence. Plainly, when the (τ) line passes through the cusp, the three points of tangency coincide and we have a line of six-point contact, mentioned above. The (σ) congruence contains ∞^2 bitangents to the focal surface, Fig. 3. The points F_1 and F_2 are the focal points, L_1 and L_2 are the limiting points. F_1 and F_2 may coincide as in figure 4, and we have a line of four point contact with the focal surface. The locus of the ∞^1 lines of four-point contact is a surface of order eight.* The ∞^1 lines of triple contact

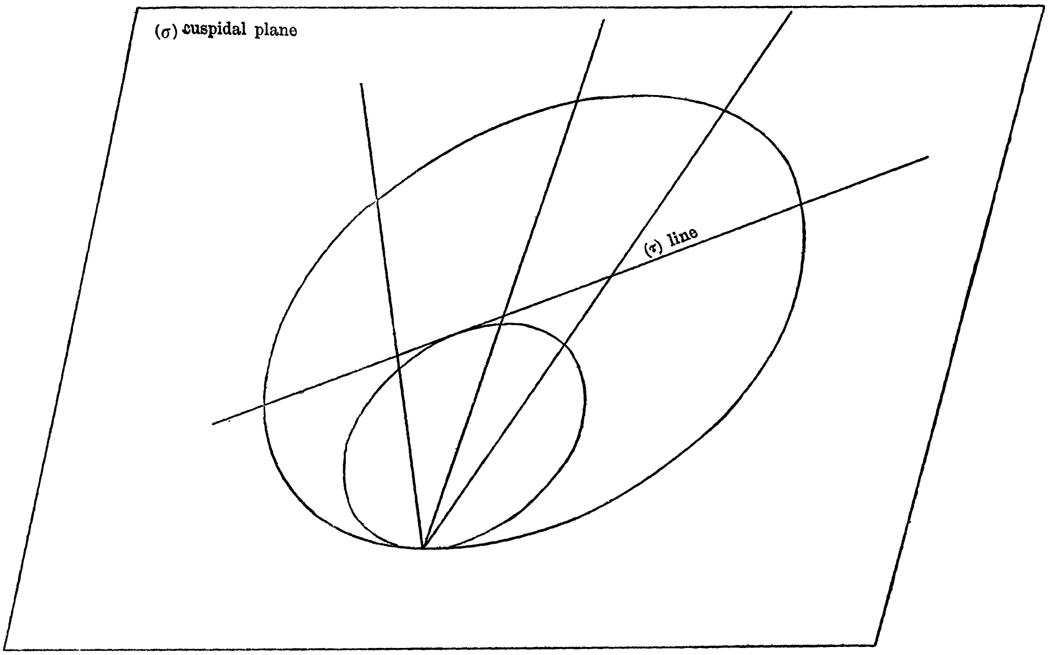

FIG. 2.

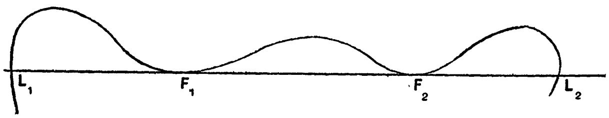

FIG. 3.

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*Study, *l. c.*, p. 485.

with the focal surface, which belong to the (σ) congruence, are the (σ) lines in the (τ) cuspidal planes, Fig. 5.

The vertical planes touch the focal surface in one circle and cut it in another circle. The infinite points on the one circle are the cusps lying in the plane. The tangents at the cusps are the lines of (σ) cut from the cuspidal planes, hence the two circles are concentric. Any plane section of the focal surface has the following Pluecker numbers:

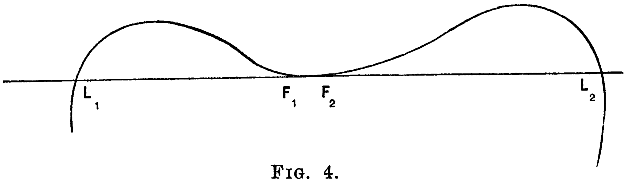

FIG. 4.

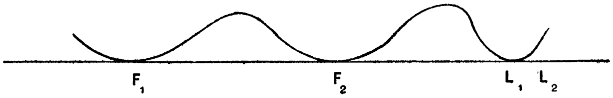

FIG. 5.

$$m=6,\quad n=12,\quad \delta=0,\quad k=6,\quad i=24,\quad \tau=27,\quad p=4.$$

For a tangent plane the numbers are

$$m=6,\quad n=10,\quad \delta=1,\quad k=6,\quad i=18,\quad \tau=15,\quad p=3.$$

From an arbitrary point, the tangent cone to the surface has the numbers

$$m=12,\quad n=4,\quad \delta=28,\quad k=24,\quad \tau=0,\quad i=0,\quad p=3,$$

and for a point in a singular (double) plane

$$m=10,\quad n=4,\quad \delta=16,\quad k=18,\quad \tau=1,\quad i=0,\quad p=2.$$

The remaining special cases can be easily determined.

CHAPTER II.

§3, S_5. *Surfaces in a Special Linear Complex.*

Since a (1, 1) correspondence exists between the points $\sigma_1, \sigma_2, \sigma_3$, in the plane $w=0$, and the finite lines of the congruence passing through them, any ruled surface contained in the congruence is completely defined by the relation among the σ_i, the equation of the section of the surface by the plane $w = 0$. In particular, if such a surface is contained in a linear complex

$$\Sigma a_{ik} p_{lm} = 0,$$

and we substitute the coordinates of the lines of (σ) for the p_{lm}, we obtain a cubic curve in terms of the σ_i, of the form

$$\frac{e_1 \sigma_1^2 + e_2 \sigma_2^2 + e_3 \sigma_3^2}{\sigma_1^2 + \sigma_2^2 + \sigma_3^3} = \frac{a\sigma_1 + b\sigma_2 + c\sigma_3}{a'\sigma + b'\sigma_2 + c'\sigma_3}. \tag{10}$$

Hence the image of any ruled surface contained in a linear complex is a c_3 passing through each of the four cusps. Conversely, since every such cubic curve contains five independent constants, it follows that the ∞^5 linear complexes and the ∞^5 cubic curves through the cusps are in (1, 1) correspondence. When $\Sigma a_{ik}\, a_{lm} = 0$, the complex becomes special; the ruled surface now consists of the lines of the congruence which cut the line a_{ik}. The c_3 cuts the absolute in two points apart from the cusps. The necessary and sufficient condition that the tangents to the absolute at these points should intersect on the curve is $\Sigma a_{ik}\, a_{lm} = 0$. Hence the complex is special when and only when the tangents to the absolute at the non-singular points in which it intersects the c_3, intersect on the latter.

Given any curve of order n in the plane $w = 0$. To determine the order of the ruled surface having this curve for section in the plane $w = 0$, it is sufficient to count the number of its generators which intersect an arbitrary line p_{ik}. The latter is the axis of a special linear complex, and the image of the lines of the congruence belonging to it is a c_3. The points of c_n on c_3, not at the cusps, correspond to the lines common to both ruled surfaces, and thus determine the number of lines of the ruled surface which cut the given line. If c_n goes through the cusp C_i, r_i times, the order of the ruled surface is $3n - r_1 - r_2 - r_3 - r_4$.*

We shall first study the ruled surfaces in a special linear complex. Through each point of an arbitrary line pass three lines of the congruence, and each plane through the arbitrary line contains two lines of the congruence, therefore, a ruled surface of order five is generated. The order of the double curve on such a surface is $6 - p$, where p is the genus of the surface. The triple line pierces the plane $w = 0$ in a point P, from which the tangents to the absolute are generators on the S_5. The remaining curve of section in this plane is a cubic,

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*Study, *l. c.*, p. 463.

which passes through P, the four cusps and the points of contact of the two tangents to the absolute from P. This surface will, in general, not have a double generator, hence the double curve is of order five. The residual curve to the triple line is a conic cutting the triple line once. This type of surface is *B II* in Schwarz's classification,[*] Snyder's (23).[†] The surface can be generated by the lines joining corresponding points of a (3, 2) correspondence between a straight line and a conic having a double self-corresponding element at their point of intersection.

Through the point P pass ∞^2 finite lines, and in general, to each one of these lines corresponds an S_5. The lines of the congruence can be arranged on ∞^1 such S_5. We will consider a pencil of ∞^1 directrices in any finite plane. In order to get the ruled surfaces having such a pencil for directrices, first consider the directrix through $(a, b, 1)$ in $z = 0$ and through (λ, μ, ν) in $w = 0$. Its coordinates are

$$p_{12} = a\mu - b\lambda, \qquad p_{13} = a\nu, \qquad p_{14} = -\lambda,$$
$$p_{23} = b\nu, \qquad p_{42} = \mu, \qquad p_{34} = -\nu.$$

A general line of the congruence will cut this line when

$$(b\lambda - a\mu)\,\sigma_3 - \nu\,(p - e_3)\,\sigma_3 + a\nu\sigma_2 - \mu\,(p - e_2)\,\sigma_2 - \lambda\,(p - e_1)\,\sigma_1 - b\nu\sigma_1 = 0. \qquad (11)$$

An equation of the form

$$pa + qb + r = 0 \qquad (p,\ q,\ r \text{ constants})$$

represents a line in the plane $z = 0$. If by means of this equation b be eliminated from the general cubic, a linear pencil of cubics

$$c_3 + ac_3' = 0$$

results, which has nine basis points; P, the four cusps C_i, the two points of contact of the tangents from P to the absolute, T_i, and two others, through each of which passes a line of the congruence, common to all the ∞^1 ruled surfaces whose directrices are lines of the plane pencil. They are the lines of the congruence in the plane of the pencil. Hence to find the basis points of the pencil of cubics, find the two points in which the line of intersection of the plane $w = 0$ and the plane of the pencil of the directrices cuts any cubic of the pencil.

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[*] "Ueber die geradlinigen Flaechen fuenften Grades." *Crelle*, vol. 67, pp. 23–57, (1867).

[†] Bulletin, vol. 8, pp. 293–296.

The point of intersection of the two lines of (σ) in the plane of the pencil of directrices is a point on the double conic, therefore, the double conics of all the ∞^1 ruled surfaces go through a common point. All the S_5 have five generators in common, two in the plane of the pencil and three through the vertex of the pencil.

If the plane of the pencil P, $pa + qb + r = 0$, does not go through a cusp nor a vertex, each S_5 has just one line belonging to each cuspidal pencil and to each vertical pencil, namely, the line joining the cusp or vertex to the point in which the directrix pierces its plane. Similarly, if the plane is not tangent to any of the conics in the (τ) cuspidal planes, no lines in these planes will belong to more than one S_5 of the pencil. The only lines which belong to more than one S_5 are the finite lines of S_5 through P, the two tangents to the absolute through P, and the two lines of (σ) in the plane of the pencil of directrices. Given P and the line of the pencil in the plane $w = 0$, any S_5 is uniquely determined when one line of it is known. From the formula, the S_5 corresponding to the infinite line through P breaks up into a cubic surface and the plane $w = 0$, counted twice. Except for particular positions of the plane, the other S_5 will not be factorable.

§4. *Other Congruences having same Focal Surface.*

Through each point of an arbitrary line l, in the plane $w = 0$, passes one finite line of the (σ) congruence. These lines of the congruence form a cylindroid whose double directrix is a finite line passing through the pole of l with respect to the absolute. If the line l passes through a cusp, the surface of the third order breaks down; the factors being a quadric and the cuspidal plane at the cusp. If l passes through two cusps, the surface consists of the two corresponding cuspidal planes and the vertical plane passing through l. To return to the case in which the line l passes through one cusp, say, $\sqrt{e_2 - e_3} : \sqrt{e_3 - e_1} : \sqrt{e_1 - e_2}$; we may write the equation of this line in the form

$$\sqrt{e_1 - e_2}\,(\sigma_1 - \lambda\sigma_2) = \sigma_3\,(\sqrt{e_2 - e_3} - \lambda\,\sqrt{e_3 - e_1}). \tag{12}$$

The lines of (σ) which cut l form a paraboloid. One line of the paraboloid must lie in the plane at infinity. It must be the tangent at k, the point in which l cuts the absolute again. All the lines of the second generation must cut this tangent at k. By letting λ take all the values, a new $(3, 2)$ congruence (π_1) is determined. l is itself a line of the new congruence, therefore the plane at infinity is a singular plane of the first order for the new congruence. In the same way we get three other congruences (π_2), (π_3), (π_4) for each of the three remaining cusps. The cusps are the vertices of the four pencils. Let P be any point in the plane at infinity. Through P can be drawn two tangents to the absolute t_1, t_2. Let t_i touch the absolute at T_i. Let C_1 be one of the cusps. $T_1\,C_1$ determines a paraboloid. One system of lines on this paraboloid belongs to the (σ) congruence. t_1 belongs to (σ), but every generator that cuts t_1 belongs to (π_1). In particular, through P passes a generator g_1 of the paraboloid λ_1. Similarly, through P passes a generator g_2 of the paraboloid λ_2, determined by the line $T_2\,C_1$. The third line of (π_1) is the line PC_1. The plane determined by g_1, g_2, cannot contain a third line of the congruence, since it would be a tangent plane to a third (σ) paraboloid and therefore contain a generator of the other system, but the other system belongs to (σ), hence the plane would contain three lines of (σ) which is impossible. If g_1 be chosen for a directrix of a ruled surface in (σ), the parabeloid λ_1 must be a factor of the S_5. The residual surface must be a cubic S_3 having g_1 as a double directrix, since through each point of g_1 pass three lines of (σ), one of which belongs to λ_1. The trace of the paraboloid λ_1 in the plane at infinity is t_1 and $T_1\,C_1$. Two generators of the cubic surface pass through P, namely, t_2 and the finite line of (σ) through P. Since the section of every ruled surface of (σ), in the plane at infinity, which belongs to a special linear complex must pass through all four cusps, it follows that the section made on S_3 by this plane consists of t_2 and a conic passing through P, the three remaining cusps and T_1. Similarly, if g_2 be chosen for the directrix, another S_3 will result. The two conics will each pass through P, C_2, C_3, C_4. The new S_3 will contain t_1 instead of t_2. This configuration exists for every point of the plane at infinity. Moreover a similar system of paraboloids and of S_3 exists for the same point P with regard to the congruences (π_2), (π_3), (π_4). The association of the S_3 with the conics is given by the formula $3n - r_1 - r_2 - r_3 - r_4$.

Since g_i is tangent to the focal surface in two points, it cuts the same surface in two other points. These points are pinch points on the S_3. The S_3 are not cylindroids.

From the manner in which the paraboloids are generated, it is seen that the lines of the congruence can be arranged on either system of ∞^1 such surfaces. Hence, the common focal surface of the two reciprocal or complementary congruences (σ) and (τ) is also the focal surface of the four other congruences (π_1), (π_2), (π_3), (π_4), which are of order three and class two.

The (σ) and (τ) cuspidal planes at C_i are singular planes of the second order in (π_i).

A plane section of the focal surface is a curve of order six, and has six cusps, but no double points. From Pluecker's numbers, we saw, it has twenty-seven bitangents. In this plane are two lines of each of the six congruences; they are all bitangents. The other fifteen are lines of intersection with the fifteen double planes. Similarly, the twenty-eight double edges of a cone from any point are the 3 × 6 lines of the six congruences passing through the point and the six lines to the vertices, and four to the cusps.

§5. S_5 *With Triple Directrix and Double Generator.*

The directrix l of an S_5 will cut the focal surface in six points. Since each of the six points is a focal point for a line of the congruence, it is a pinch point on the S_5. Through l can be drawn four planes which are tangent to the focal surface. Since each of these four planes is a focal plane for a line of the congruence, the focal point in this plane is also a pinch point on the S_5, making in all, ten pinch points on the S_5, which agrees with Lueroth's formula. The four pinch points in the tangent planes through l are on the double conic. Two of the pinch points on l coincide when l becomes tangent to the focal surface, but when l is such a tangent, two of the tangent planes to the focal surface through l coincide. Therefore, two of the generators of the S_5 which pass through the coincident pinch points of l have also a common point on the double conic, or we have a double generator on the S_5. Hence, among the ∞^2 lines through an arbitrary point P, are ∞^1 tangent lines to the focal surface, to each of which corresponds a ruled surface of order five having a double generator. In general, the cone of tangents to the focal surface from any point P is of order twelve, it has twenty-four cuspidal edges and twenty-eight double edges. When P is in a singular (double) plane, we saw that the cone was of order ten. For certain positions the cone may be of lower order. On those ruled surfaces S_5, which have double generators the corresponding cubics in the plane $w = 0$ have nodes. The c_3 corresponding to a bundle of lines form a net; their basis points are, when the point P is taken in the plane $w = 0$, P, C_1, C_2, C_3, C_4, T_1, T_2. The locus of the node is the locus of the trace of the double generator in the plane $w = 0$. We may obtain the equation of this locus by forming the Jacobian of the net of curves represented by the equation (11). Equation (11) may be written in the form

$$a\,(\nu\sigma_2 - \mu\sigma_3)\,\Sigma + b\,(\lambda\sigma_3 - \nu\sigma_1)\,\Sigma + (\nu e_3\sigma_3 + \mu e_2\sigma_2 + \lambda e_1\sigma_1)\,\Sigma \\ - (\nu\sigma_3 + \mu\sigma_2 + \lambda\sigma_1)\,(e_1\sigma_1^2 + e_2\sigma_2^2 + e_3\sigma_3^2) = 0, \qquad (13)$$

where $\Sigma = \sigma_1^2 + \sigma_2^2 + \sigma_3^2$. Its Jacobian is

$$\begin{vmatrix} 2\sigma_1(\nu\sigma_2 - \mu\sigma_3) & \nu\Sigma + 2\sigma_2(\nu\sigma_2 - \mu\sigma_3) & -\mu\Sigma + 2\sigma_3(\nu\sigma_2 - \mu\sigma_3) \\ -\nu\Sigma + 2\sigma_1(\lambda\sigma_3 - \nu\sigma_1) & 2\sigma_2(\lambda\sigma_3 - \nu\sigma_1) & \lambda\Sigma + 2\sigma_3(\lambda\sigma_3 - \nu\sigma_1) \\ (\qquad\qquad) & (\qquad\qquad) & (\qquad\qquad) \end{vmatrix}$$

This locus consists of the absolute and a quartic curve having a double point at P, passing through the cusps, and touching the absolute at T_1, T_2.

We may verify these results with some special cases. Let P be taken at $\sigma_1 = 0$, $\sigma_2 = 0$. Through this point pass ∞^2 lines, all parallel to the principal ray $x = 0$, $y = 0$. Any one of these lines $x + aw = 0$, $y + bw = 0$, is the axis of a special complex. The condition that this line cuts the line

$$\begin{gathered} \sigma_3 y - \sigma_2 z - (p - e_1)\sigma_1 w = 0 \\ -\sigma_3 x + \sigma_1 z - (p - e_2)\sigma_2 w = 0 \end{gathered}$$

is $a\sigma_2 + (p - e_3)\sigma_3 - b\sigma_1 = 0$, or solving for p

$$\frac{b\sigma_1 - a\sigma_2 + e_3\sigma_3}{\sigma_3} = \frac{e_1\sigma_1^2 + e_2\sigma_2^2 + e_3\sigma_3^2}{\sigma_1^2 + \sigma_2^2 + \sigma_3^2} \tag{14}$$

This cubic is the trace in the plane $w = 0$, of the S_5 having $x + aw = 0$, $y + bw = 0$, for the triple line. It passes through $(0, 0, 1)$ and cuts the absolute at $(i, \pm 1, 0)$. The tangents to the absolute at these points pass through $(0, 0, 1)$. In general this cubic is of genus one. On putting $\sigma_1 = m\sigma_2$ in the equation and letting $\sigma_3 = 1$, we have

$$(mb - a)(m^2 + 1)\sigma_2^2 + [(e_3 - e_1)m^2 + (e_3 - e_2)]\sigma_2 + mb - a = 0. \tag{15}$$

The discriminant of this equation is

$$\begin{aligned} [4b^2 - (e_3 - e_1)^2]m^4 - 8abm^3 + [4b^2 + 4a^2 - 2(e_3 - e_1)(e_3 - e_2)]m \\ - 8abm + 4a^2 - (e_3 - e_2)^2 = 0. \end{aligned} \tag{16}$$

For a given value of m, the two roots of (15) define the points in which the line $\sigma_1 = m\sigma_2$ cuts the cubic. When these two points coincide, the line is a tangent, hence the directions of the four tangents to c_3 from P are the roots of equation (16). If equation (16) has a double root, c_3 has a node, *i. e.* c_3 has a node when the m-discriminant of (16) vanishes. This is an expression of order twelve in a and b, which agrees with the result obtained above. The cubic now becomes

$$a\sigma_2(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) - b\sigma_1(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) + \sigma_3[(e_1 - e_3)\sigma_1^2 + (e_2 - e_3)\sigma_2^2] = 0. \tag{17}$$

The Jacobian is

$$\begin{vmatrix} 2\sigma_1\sigma_2 & -\Sigma - 2\sigma_1^2 & 2\sigma_1\sigma_3(e_1 - e_3) \\ \Sigma + 2\sigma_2^2 & -2\sigma_1\sigma_2 & 2\sigma_2\sigma_3(e_2 - e_3) \\ 2\sigma_2\sigma_3 & -2\sigma_1\sigma_3 & (e_1 - e_3)\sigma_1^2 + (e_2 - e_3)\sigma_2^2 \end{vmatrix}$$

or

$$\Sigma\,[\sigma_1^2(e_1 - e_3) + \sigma_2^2(e_2 - e_3)]\,[(\sigma_1^2 + \sigma_2^2 - \sigma_3^2)] = 0.$$

Thus the Jacobian breaks up into the absolute, the two lines through P and the cusps, and another conic touching the absolute at the points of contact of the tangents; namely $\sigma_1^2 + \sigma_2^2 - \sigma_3^2 = 0$. The lines of (σ), which cut a line joining two cusps, are the lines of a pencil whose vertex is a vertex. The lines of (σ) which cut the absolute, are its tangents, in the plane $w = 0$; hence, the locus of double rays, which are finite lines not belonging to a vertical pencil, must have the conic $\sigma_1^2 + \sigma_2^2 - \sigma_3^2 = 0$ for its trace in the plane $w = 0$. The ruled surface formed by double generators is of order six. It is rational since a simple conic lies on it. The two tangents from P to the absolute are each double generators. The residual double curve is of order eight.

If we put the coefficients of the two highest powers of m in equation (16) equal to zero, the equation has a double root at infinity. The corresponding value of σ_2 in (15) is $\sigma_2 = 0$. From (14) the corresponding values of σ_1 are $\sigma_1(\sigma_1 \pm 1)^2 = 0$. But $\sigma_1 = 0$ is at the origin, hence the coordinates of the double points are $\sigma_1 = \pm 1$, $\sigma_2 = 0$, $\sigma_3 = 1$. The equations of the double generator of the S_5 having the line $x = 0$, $2y = \pm (e_1 - e_3)w$, for triple line, are therefore

$$y \pm (p - e_1)w = 0 \quad x \pm z = 0; \text{ but}$$
$$p = (e_3 - e_1)\,/\,2, \text{ therefore } 2y = \pm (e_1 - e_3)w.$$

Between

$$\frac{(e_1 - e_3)\sigma_1 + 2e_3\sigma_3}{2\sigma_3} = \frac{e_1\sigma_1^2 + e_2\sigma_2^2 + e_3\sigma_3^2}{\sigma_1^2 + \sigma_2^2 + \sigma_3^2} = \frac{\sigma_3 y - \sigma_2 z + e_1\sigma_1 w}{\sigma_1 w}$$
$$= \frac{-\sigma_3 x + \sigma_1 z + e_2\sigma_2 w}{\sigma_2 w} = \frac{\sigma_2 x - \sigma_1 x + e_3\sigma_3 w}{\sigma_3 w}$$

it is possible to eliminate the σ_i rationally. The result will be an S_5 having a triple directrix, a distinct double generator and a double conic.

If we put $x = 0$, $2y + (e_1 - e_3)w = 0$, $w = 1$, in the equation of the focal surface, we have

$$(-3e_2^2 + 4z^2)^3 + 27(e_2^3 + 4e_3z^2)^2 = 0.$$

This equation has z^2 for a factor, therefore the directrix touches the focal surface. The double generator intersects the directrix at $(o, (e_3 - e_2), o, 2)$, *i. e.* at the point in which the directrix touches the focal surface. The plane containing the directrix and the nodal generator is given by the equation $2y + (e_1 - e_3) w = 0$, but this is the tangent plane to the focal surface at the point $(o, (e_3 - e_2), o, 2)$, as is easily verified.

Among the $\infty^2 S_5$, associated with the lines passing through an arbitrary point P, the $\infty^1 S_5$, whose directrices are the tangents from P to the focal surface, have three lines in common. The lines of the congruence can all be arranged on such ruled surfaces, each having a double generator.

§6. *Directrix l a Line of* (σ).

When the directrix is a line of (σ), through each of its points pass two lines of (σ) besides l itself. There can be no double curve on the locus apart from l, since from each point of such a curve, would pass two generators, each of which would cut l. That would be impossible because only two lines of (σ) lie in a plane. Every generator of an S_5 cuts three other generators, hence l must count for two generators as well as a double directrix. The surface has, therefore a four-fold line in l. It is unicursal. This type of ruled surface is a particular form of Schwarz's A I, but not mentioned, Snyder's (31). The curve on any plane section of the ruled surface containing two generators is a nodal cubic having its mode at the point of intersection of the generators (*i. e.* where l pierces the plane). Such a surface can be generated by a nodal cubic and a straight line through the node in (1, 2) correspondence without self corresponding elements. The points at which one of the variable generators coincides with l are the foci; the points at which two generators, distinct from l, coincide are limiting points. These are the pinch points on the ruled surface. In the former case l touches the focal surface, in the latter, it intersects the focal surface. The foci are the points on l which correspond to the two values of the parameter of the cubic at the node. The limiting points are the branch points of the (1, 2) correspondence. The limiting points divide the line l, considered as a directrix of a surface, into real and imaginary segments. If the foci are in the real segment, the corresponding cubic in the plane $w = 0$ is crunodal. If the foci are in the imaginary segment the cubic is acnodal. If l touches the focal surface in a point of the curve of order sixteen, mentioned by Study (*l. c.* p. 485), the two foci coincide, and the cubic curve in the plane $w = 0$, has a cusp.

The tangents to the absolute at the variable points of intersection of the c_3 with the absolute, are the generators of the S_5 in the plane $w=0$. These tangents intersect at the node of the c_3.

If the point P describes any locus in the plane $w=0$, the corresponding S_5 will comprise the entire (σ) congruence, but each line will belong to several surfaces. If the locus is of order n going r times through C_i, each line of (σ) will belong to $3n-r_1-r_2-r_3-r_4$ different ruled surfaces.

§7. *Degraded S_5.*

Some interesting cases of degraded S_5 may be obtained by letting the directrix l take special positions. When l lies in a singular plane or passes through a singular point, the corresponding S_5 is reducible. In particular, the discriminant curve, $I^3-27J^2=0$, of equation (16), in the plane $z=0$, will contain the traces of the vertical planes through (x, y) in the plane, as factors, each counted twice. Any line from a point of either of these lines to $P\equiv(0, 0, 1, 0)$, will be the directrix of a degraded S_5, since the vertical plane will appear as a factor.

The vertical planes cut the plane $w=0$ in the lines joining two cusps. These lines are therefore factors of the respective associated cubic curves. The other factor is a conic passing through the remaining two cusps, and through the points of tangency of the tangents from P to the absolute. Let Q be a point on the trace of a vertical plane in $z=0$. The line PQ determines a ruled surface and therefore determines a conic in the plane $w=0$. The conics and points Q are therefore projective and the corresponding pair of double points (intersection of the line and the conic) are in involution. The ruled surface of the fourth order determined by this conic, has PQ for a double line. It has also a double conic cutting it; it is always unicursal. This surface can be generated by the points of a conic and a line cutting it, in (2, 2) correspondence, the point of intersection being a double self-corresponding element. It is Cayley's VII.

For certain points on the discriminant curve, the corresponding cubic curve will have a cusp; these points are at the intersection of $I=0$, and $J=0$. They are also cusps on the discriminant curve. For a finite number of points the discriminant will be a perfect square. This means that the cubic curve will have two double points, hence become reducible.

The lines of the congruence can be arranged on the ∞^1 S_4, whose directrices lie in a vertical plane and pass through an arbitrary point in this plane.

If the directrix be an arbitrary line through a vertex, the vertical plane occurs as a factor, and we have remaining an S_4. This surface has a triple directrix, and no simple directrix. This is Cayley's IX. In this case the lines of the congruence can be arranged again on ∞^1 S_4 whose directrices are a plane pencil of lines through the vertex. Every line of the congruence belongs to just one such surface, except the focal axis passing through the vertex. It belongs to every ruled surface of the system. Two lines of the vertical pencil whose vertex lies on l, belong to S_4, but a different pair for each line l of the pencil. No line of the pencil can belong to two ruled surfaces, because they are determined from the points of intersection of the trace of the plane in the plane $w = 0$ and the conic, the trace of the S_4 in the plane $w = 0$. All such conics, intersect in C_1, C_2, T_1, T_2, hence they can have no other points in common.

When the directrix l lies in the plane $w = 0$, a conic C_2 through l, the pole of l with respect to the absolute, and the four cusps, is uniquely determined. l and C_2 make a degraded c_3 corresponding to a degraded ruled surface of order five. l is the simple directrix of a cylindroid, as has been pointed out, whose double line is the finite line of (τ) passing through L. Thus to any line l, in the plane $w = 0$, is a definite C_2, but conversely if C_2 be given, there are ∞^1 lines l associated with it, any one of which, will, with C_2, determine a degraded S_5. The simple directrices of these cylindroids envelop a conic in the plane $w = 0$, which is the polar conic of C_2 with respect to the absolute. C_2 is also the trace of the ∞^1 double directrices of the family of cylindroids. These lines form the (τ) system of the parameter surface through L. In this way every line of the (σ) congruence is counted twice. By interchanging the (σ) and the (τ), the other system of the parameter surface would be the double directrices; the simple directrices would be the same as before.

Now let l become tangent to the absolute; the polar of l becomes the point of contact, *i. e.* the simple directrix and the double directrix coincide and we have an S_3 known as Cayley's S_3. A plane through l contains one generator of each of the S_3 ((σ) and (τ)) corresponding to l. The locus of the point of intersection of the generators is a plane cubic curve having a node at the point of tangency of l and the absolute. The S_3 of the (σ) congruence intersects the S_3 of the (τ) congruence in a plane cubic curve.

Besides, there are a finite number of other ruled surfaces. If l passes through two vertices, the two vertical planes associated with these vertices are factors of the S_5. If l is a principal ray, it is a line of (τ) as well as (σ), and the lines of (σ) which cut it from a cylindroid whose directrix in the plane $w = 0$ is the side of the self-polar triangle opposite the vertex which is pierced by the pincipal ray. If l is a focal axis, the two cuspidal planes, which intersect on this focal axis, are factors of the S_5.

§8. *Other Loci.*

Besides the surfaces which comprise the arrangement of the lines of the congruence, some others present themselves, which are worthy of mention.

From §5, we see that associated with any point P in the plane at infinity is a ruled surface formed of the double generators. The trace of this surface, in the plane at infinity, is the Jacobian of the cubic and the two tangents from P to the absolute. The Jacobian is of order four and passes once through each cusp. Therefore the surface is of order eight. It is of genus two. The total nodal curve is of order nineteen, or seventeen besides the double generators (tangents to the absolute from P). The finite line of (σ) through P is a generator.

In every pencil of planes, having a finite axis, is just one plane such that the lines of (σ) lying in this plane intersect on the axis. This is the point in which the axis is cut by the double conic belonging to the S_5 determined by the given axis as directrix. Corresponding to the bundle of lines through a given point we would have ∞^2 such points of intersection forming a surface. To find the form of this locus, take any directrix line l through P and let it cut the double conic in Q. Pass a plane π through l and a (σ) line σ_p, through P. π contains another line g_1 of (σ). These two lines intersect at R on the double conic. Hence, the double conics for each directrix l in the plane π must pass through R. Moreover, through R passes a third line g_2 of (σ). The pencil P in the plane (σ_p, g_2) will define ∞^1 S_5 whose double conics pass through R. Then to each point of σ_p corresponds two sets, each containing an infinity of conics. Hence the three lines of (σ) through P are double lines on the surface. The point P is a triple point. Each line through P pierces the surface in one other point apart from P, hence the locus of the point is a surface of order four, having three double lines through R, hence, a Steiner surface of class three. When P lies in a singular (double) plane, for example, the plane at infinity, this plane must be a factor of the surface, since two (σ) lines through any point in this plane lie in the plane. In this case the surface is of order three, having the finite line of (σ) through P for a double line, hence it is a ruled surface. The tangents to the absolute from P, are simple lines on the S_3, and the simple directrix lies in the plane at infinity. It is the polar of P with regard to the absolute, hence the surface is a cylindroid.

Every point of every double conic associated with P lies on the Steiner surface associated with P. When a plane cuts a conic from a Steiner surface, it must be a tangent plane, hence the planes of the double conics envelop the same Steiner surface.

Through each point of the double conic, belonging to the S_5, pass three lines of the congruence. Two of these lines belong to the S_5. The third line generates a new surface of order five, on which the conic is simple. It belongs to another linear complex, usually not special, and contains a double generator.

CHAPTER III.

Surfaces in a Non-Special Linear Complex.

In the former chapter, the ruled surfaces which composed the arrangement of the lines of the congruence, belonged to a special linear complex. In this chapter we shall investigate the surfaces formed by the lines of the congruence, when the complex to which they belong is non-special. The cubic curve in the plane at infinity will not pass through the point of intersection of the tangents to the absolute at the variable points of intersection of this c_3 with the absolute (Chapter II, §3). The new surfaces cannot have a rectilinear directrix.

§9. c_3 *in Plane at Infinity.*

The line coordinates of a line joining the point $(-a, -b, 1)$ in the plane $z = 0$ and the point (λ, μ, ν), in the plane $w = 0$, are

$$\begin{array}{lll} p_{12} = -a\mu + b\lambda, & p_{13} = -a\nu, & p_{14} = -\lambda, \\ p_{23} = -b\nu, & p_{42} = \mu, & p_{34} = -\nu. \end{array}$$

This line is the axis of a special linear complex whose equation is

$$A \equiv \nu p_{12} + (a\mu - b\lambda)p_{34} - \mu p_{13} + a\nu p_{42} + b\nu p_{14} + \lambda p_{23} = 0.$$

The pencil of coaxial complexes, having the same line for axis is

$$A + k(\nu p_{34} - \mu p_{42} + \lambda p_{14}) = 0.$$

The corresponding pencil of cubic curves in the plane $w=0$ is obtained by substituting for p_{ik}, the coordinates of the axis:

$$\nu(p-e_3)\sigma_3-(a\mu-b\lambda+k\nu)\sigma_3+\mu(p-e_2)\sigma_2+(a\nu-k\mu)\sigma_2 \\ -(b\nu+k\lambda)\sigma_1+\lambda(p-e_1)\sigma_1=0.$$

and eliminating p. We have then

$$(\lambda\sigma_1+\mu\sigma_2+\nu\sigma_3)\frac{e_1\sigma_1^2+e_2\sigma_2^2+e_3\sigma_3^2}{\sigma_1^2+\sigma_2^2+\sigma_3^2}=\nu e_3\sigma_3+(a\mu-t\lambda+k\nu)\sigma_3+\mu e_2\sigma_2 \\ -(a\nu-k\mu)\sigma_2+(b\nu+k\lambda)\sigma_1+\lambda e_1\sigma_1=0. \qquad (18)$$

This is a linear system of ∞^3 c_3, obtained by varying a, b, k. All the curves of the system pass through C_i, the four cusps, and T_1, T_2, the points of contact of the tangents to the absolute from $P\equiv(\lambda,\mu,\nu)$ in the plane $w=0$. If we regard a, b as constant, we obtain ∞^1 coaxial complexes:

$$(\lambda\sigma_1+\mu\sigma_2+\nu\sigma_3)(e_1\sigma_1^2+e_2\sigma_2^2+e_3\sigma_3^2)-(\lambda e_1\sigma_1+\mu e_2\sigma_2+\nu e_3\sigma_3)(\sigma_1^2+\sigma_2^2+\sigma_3^2) \\ -(a\mu-b\lambda)\sigma_3(\sigma_1^2+\sigma_2^2+\sigma_3^2)+(a\nu\sigma_2-b\nu\sigma_1)(\sigma_1^2+\sigma_2^2+\sigma_3^2) \\ =k(\lambda\sigma_1+\mu\sigma_2+\nu\sigma_3)(\sigma_1^2+\sigma_2^2+\sigma_3^2). \qquad (19)$$

When k is equal to zero, we have the same cubic curve that we studied in connection with the special complex. The basis points of the system of cubic curves are the four cusps, the intersection of $\sigma_1^2+\sigma_2^2+\sigma_3^2=0$, and $e_1\sigma_1^2+e_2\sigma_2^2+e_3\sigma_3^2=0$; the points of T_1, T_2, the intersection of $\sigma_1^2+\sigma_2^2+\sigma_3^2=0$ and $\lambda\sigma_1+\mu\sigma_2+\nu\sigma_3=0$, each counted twice, since all the c_3 touch each other at T_1, T_2; the point determined by the intersection of $(\lambda e_1+b\nu)\sigma_1+(\mu e_2-a\nu)\sigma_2+(\nu e_3+a\mu-b\lambda)\sigma_3=0$ and $\lambda\sigma_1+\mu\sigma_2+\nu\sigma_3=0$. When $k=\infty$, the corresponding c_3 factors into the absolute and the polar line of P with regard to the absolute; this factorable c_3 has two nodes. The pencil $c_3+kc_3'=0$ has ten other nodal cubics.

§10. S_5 *in Congruence.*

To each cubic curve in the plane at infinity corresponds an S_5. The cubic will not, in general, have a node. Hence the genus of the ruled surface is one, and the nodal curve is a space curve of order five. This type is Schwarz's BI, Snyder's (22). Every generator cuts the nodal curve three times. From an arbitrary point upon it, the twisted quintic will project into a quartic cone, and the two generators from the point of projection will be double edges upon it, hence the quintic curve is of genus one. An arbitrary line cuts an S_5 in five points. The axis of the complex cuts two generators in the plane $w=0$. The lines of a non-special linear complex which cut the axis, must cut it at right angles, hence, the finite lines of the S_5 which cut the axis must pierce the plane $w = 0$ on the line $\lambda\sigma_1 + \mu\sigma_2 + \nu\sigma_3 = 0$, but there is only one such point on the cubic curve apart from T_1, T_2. This is evidently true for every arbitrary line in space, hence, only one finite line of the congruence cuts an arbitrary line (not belonging to (τ)) at right angles.

The tangent lines to the absolute from P, each count twice, as lines of the congruence intersecting the axis of the complex, hence the axis has four point contact with S_5 at $(\lambda, \mu, \nu, 0)$. The lines which cut the axis are independent of k; they are common to all the complexes of the coaxial pencil. Each of the nodal curves of the surfaces corresponding to the ∞^1 values of k, will pass through P. The lines of the congruence can be arranged on these surfaces, such that every line will belong to one complex.

The nodal quintic curves through P will generate a surface, which is a factor of the k-envelope of the family of S_5.

The two generators of the S_5 in the plane at infinity, cut the cubic in three points. One of these points is T_i, the point of tangency to the absolute. The nodal curves of the S_5 pass through T_i. The two generators in the plane $w = 0$ lie on the surface formed by the nodal quintics of the various S_5. Similarly the finite line cutting the axis lies on this surface.

Every ruled surface contained in a linear complex has an algebraic asymptotic line whose order is equal to the class of an arbitrary plane section of the surface.* In our case the plane section is a quintic having five double points, its class is therefore ten. The number of pinch points on the surface is ten. They are included among the intersections of the nodal curve and the focal surface.

§11. S_5 *Having Double Generator.*

When the cubic curve in the plane at infinity has a node, the S_5 must have a double generator, and consequently only six pinch points. All the pinch points are on the focal surface, hence, when the ruled surface has a double generator, the nodal curve must touch the focal surface twice, namely, at the foci of the double generator.

The genus of the ruled surface is now zero. The nodal curve contains the double generator as a factor; since the double curve has a triple point (Schwarz, *l. c.*, p. 12), the twisted quintic has a node, therefore, it is of genus zero.

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* Picard, "Mémoire sur une application de la théorie des complexes linéaires à l' étude des surfaces et des courbes gauches." Annales de l'Ecole Normale, 1877. Snyder, Bulletin Amer. Math. Society, vol. 5, p. 343 ff.

A plane section of the S_5 has six double points, from which the order of the complex asymptotic line is eight.

The cubic curve associated with P, and with parameter k may be written

$$\begin{aligned} a(\mu\sigma_3 - \nu\sigma_2)(\sigma_1^2+\sigma_2^2+\sigma_3^2) + b(\nu\sigma_1 - \lambda\sigma_3)(\sigma_1^2+\sigma_2^2+\sigma_3^2) \\ - (\lambda\sigma_1+\mu\sigma_2+\nu\sigma_3)(e_1\sigma_1^2+e_2\sigma_2^2+e_3\sigma_3^2) \\ + (\lambda e_1\sigma_1+\mu e_2\sigma_2+\nu e_3\sigma_3)(\sigma_1^2+\sigma_2^2+\sigma_3^2) \\ + k(\lambda\sigma_1+\mu\sigma_2+\nu\sigma_3)(\sigma_1^2+\sigma_2^2+\sigma_3^2) = 0. \end{aligned}$$

The coefficients of a and b are the same as they were for the system of cubics in the special linear complex, hence the Jacobian of the system contains the absolute as a factor. It is linear in k and passes through the basis points of the system of cubics. Any finite line of the congruence is a double generator on a ruled surface of order five belonging to a complex whose axis passes through P, and it belongs to just one such complex, since the Jacobian, which is the trace of the double generators in $w = 0$ is linear in k; and k is therefore uniquely determined by the double generator. The coefficient of k in the equation of the pencil of Jacobians is $3(\lambda^2+\mu^2+\nu^2)(\sigma_1^2+\sigma_2^2+\sigma_3^2)^2$. Therefore the curves touch each other at the points T_1, T_2. They do not pass through P. The cusps and T_i are all basis points.

§12. *Degraded S_5.*

By a proper choice of the axis of the complex, the ruled surfaces may become reducible. If a plane, which contains a pencil of lines of the congruence, belongs to the complex, then the S_5 is composed of this plane and a ruled surface of order four. The nodal curve on such a surface is a twisted cubic. The type of this surface is Salmon's VI, Cayley's X. An example of the above mentioned surfaces may be obtained by taking for the axis of the coaxial system of complexes, the line through a vertex and perpendicular to the vertical plane. The vertical plane will now belong to every complex of the coaxial system, and we have ∞^1 S_4 which comprise the lines of the congruence. These surfaces cannot have a double generator. If the vertical plane belongs to the complex, and the vertex is not on the axis, we have a single surface of the fourth order as described above.

In a similar way, if two plane pencils of the congruence belong to the complex, the S_5 will be composed of these two planes and a ruled surface of order three. In order that two such planes belong to the coaxial system of complexes, it is necessary that the vertices of the pencils of lines lie on the axis of the coaxial system and that the planes of the two pencils be perpendicular to this axis. This will not, in general, be true for our configuration. However, a single surface of this kind may be formed when a principal ray is a line of the complex and the vertical planes belong to the same complex.

If a line of (τ) be chosen as axis of the complex, the cylindroid, whose directrices are the axis of the complex and the polar of P with regard to the absolute, is common to all the complexes of the coaxial system. Therefore the system (∞^1) hyperboloids comprise the lines of the congruence. The line coordinates for the point (λ, μ, ν) in the plane $w = 0$ are

$$\begin{aligned} p_{12} &= -(p' - e_3)\nu, & p_{13} &= (p' - e_2)\mu, & p_{14} &= -\lambda, \\ p_{23} &= -(p' - e_1)\lambda, & p_{42} &= \mu, & p_{34} &= -\nu. \end{aligned}$$

The corresponding cubic in the plane $w = 0$ is, when

$$p = \frac{e_1\lambda^2 + e_2\mu^2 + e_3\nu^2}{\lambda^2 + \mu^2 + \nu^2}$$

$$(\lambda\sigma_1 + \mu\sigma_2 + \nu\sigma_3)(p - p' + k) = 0.$$

The factors of this equation are the polar of P with regard to the absolute and a conic passing through the four cusps. When $k = 0$, we have the relation $p = p'$, which is the condition that a line of (σ) cut a line of (τ).

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